Indian Statistical Institute
Backpaper Examination 2003-2004
M.Math I Year
Differential Geometry

Time: 3 hrs Max. Marks : 50

Answer all six questions

Possibly Useful Formula:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle$$

1. Let c be a curve with domain $(0,1)$ and image in the $x-z$ plane $(x>0)$. Let $M \subset \mathbb{R}^3$ be the two-dimensional manifold obtained by rotating c around the z-axis. Let g be the metric induced on M from $\mathbb{R}^3$. Assume that $c(u) = (r(u), z(u))$ where $r'(u)^2 + z'(u)^2 = 1$.

   a) Give an explicit description of charts for M.

   b) Prove that there are isometries of (M, g) other than the identity map.

   c) Calculate the sectional curvature of $(M,\ g)$ at any point in terms of $r(u)$ and $z(u)$.
   [11]

2. Let (N, g) be a compact, oriented Riemannian manifold with boundary. Prove that there exists a non-vanishing normal vector field on ∂N. (A vector field X is "normal" to ∂N if $< X, v >= 0$ for every $v \in T_p\partial N$ and any $p \in \partial N$). [6]

3. Let M be a compact oriented Riemannian n- manifold.

   a) Prove that if ω is any $(n-1)$-form on M, then $d\omega_p = 0$ at some p. [5]

   b) Let $\gamma : [0,1] \to M$ be a smooth curve, $\{e_1(t), .., e_n(t)\}$ be a set of parallel vector fields such that $\{e_1(0), .., e_n(0)\}$ is a positively oriented orthonormal set. Prove that $\{e_1(t), .., e_n(t)\}$ is positively oriented for every t. [5]

4. Let (M, g) be a Riemannian manifold, Z an embedded submanifold and p a point in Z. Let h denote the induced metric on Z, ∇^g and ∇^h denote the Levi-Civita connections associated to g and h. If X and Y are two vector fields tangent to Z,

   a) Prove that there exist extensions $\tilde{X}$ and $\tilde{Y}$ of X and Y to an open subset of M containg p. [4]

b) Prove that

$$\nabla^h_X Y = (\nabla^g_{\tilde{X}} \tilde{Y})^T.$$

Here v^T, for any $v \in T_pM$, denotes the orthonormal projection of v into $T_pZ \subset T_pM$. [6]

5. Let (M, g) be a Riemannian manifold and $\gamma : [0,1] \to M$ be a smooth curve. Prove that any vector field X along γ can be realized as the variation vector field of some variation of γ. [6]

6. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $N = M_1 \times M_2$ with the metric $h = g_1 + g_2$. Let X_i, $i = 1, 2$ be a vector tangent to M_i. Prove that $K_h(X_i, X_j) = 0$ if $i \neq j$ and $K_h(X_i, X_i) = K_{g_i}(X_i, X_i)$, where K_b denotes the sectional curvature of a 2-plane with respect to a metric b. [7]